

## Shear representations of beam transfer matrices

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The beam transfer matrix, often called the  $ABCD$  matrix, is one of the essential mathematical instruments in optics. It is a unimodular matrix whose determinant is 1. If all the elements are real with three independent parameters, this matrix is a  $2 \times 2$  representation of the group  $Sp(2)$ . It is shown that a real  $ABCD$  matrix can be generated by two shear transformations. It is then noted that, in para-axial lens optics, the lens and translation matrices constitute two shear transformations. It is shown that a system with an arbitrary number of lenses can be reduced to a system consisting of three lenses.

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### I. INTRODUCTION

In a recent series of papers [1,2], Han *et al.* studied possible optical devices capable of performing matrix operations of the following types:

$$T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \quad (1.1)$$

Since these matrices perform shear transformations in a two-dimensional space [3], we shall call them ‘‘shear’’ matrices. However, Han *et al.* were only interested in the ‘‘slide-rule property’’ of the shear matrices which convert multiplications into additions. The  $T$  matrix has the property

$$T_1 T_2 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 \\ 0 & 1 \end{pmatrix} \quad (1.2)$$

and the  $L$  matrix has a similar ‘‘slide-rule’’ property. This property is valid only if we restrict computations to  $T$ -type matrices or  $L$ -type matrices.

In the present paper, we study what happens to the  $ABCD$  matrix if we use both  $L$ - and  $T$ -type shear matrices, which takes the form

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (1.3)$$

where the elements  $A$ ,  $B$ ,  $C$ , and  $D$  are real numbers satisfying  $AD - BC = 1$ . Because of this condition, there are three independent parameters.

We are interested in constructing the most general form of the  $ABCD$  matrix in terms of the two shear matrices given in Eq. (1.1).  $2 \times 2$  matrices with the above property form the symplectic group  $Sp(2)$ . We are quite familiar with the conventional representation of the  $2 \times 2$  representation of the  $Sp(2)$  group. This group is like (isomorphic to)  $SU(1,1)$  which is the basic scientific language for squeezed states of

light [4]. This group is also applicable to other branches of optics, including polarization optics, interferometers, layer optics [5], and para-axial optics [6,7]. The  $Sp(2)$  symmetry can be found in many other branches of physics, including canonical transformations [3], special relativity [4], Wigner functions [4], and coupled harmonic oscillators [8].

This group, consisting of  $2 \times 2$  real matrices, also has a very rich mathematical contents. There are still hidden mathematical theorems which can be useful in managing our calculations in physics. Specifically, we use group theory to represent the most general form of the  $ABCD$  matrix in terms of the shear matrices given in Eq. (1.1). With this point in mind, we propose to write the  $2 \times 2$   $ABCD$  matrices in the form

$$TLTLT \dots \quad (1.4)$$

Since each matrix in this chain contains one parameter, there are  $N$  parameters for  $N$  matrices in the chain. On the other hand, since both  $T$  and  $L$  are real unimodular matrices, the final expression is also real and unimodular. This means that the expression contains only three independent parameters. Then we are led to the question of whether there is a shortest chain which can accommodate the most general form of the  $2 \times 2$  matrices. We shall conclude in this paper that six matrices are needed for the most general form, with three independent parameters.

We are not the first ones to study this problem. In 1985, Sudarshan *et al.* raised this question in connection with para-axial lens optics [7]. They observed that the lens and translation matrices are in the form of matrices given in Eq. (1.1). In fact, the notations  $L$  and  $T$  for the shear matrices of Eq. (1.1) are derived from the words ‘‘lens’’ and ‘‘translation,’’ respectively, in para-axial lens optics. Sudarshan *et al.* concluded that three lenses are needed for the most general form for the  $2 \times 2$  matrices for the symplectic group. Of course their lens matrices are appropriately separated by translation matrices. This will make the total number of matrices six. However, in their paper Sudarshan *et al.* stated that the calculation of each lens or translation parameter is ‘‘tedious.’’

In the present paper, we make this calculation less tedious by using a decomposition of the  $ABCD$  matrix derivable

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from Bargmann's paper [9]. As far as the number of lenses is concerned, we reach the same conclusion as that of Sudarshan *et al.* However, we complete the calculation of lens parameter for each lens and the translation parameter for each translation matrix, in terms of the three independent parameters of the  $ABCD$  matrix. In other words, we complete the calculation which Sudarshan *et al.* started in 1985.

In Sec. II, it is shown that the most general form of the  $Sp(2)$  matrices or  $ABCD$  matrices can be decomposed into one symmetric matrix and one orthogonal matrix. It is shown that the symmetric matrix can be decomposed into four shear matrices and the orthogonal matrix into three. In Sec. III, it is noted that the mathematical device developed in Sec. II is directly applicable to para-axial lens optics. It is shown that the most general lens system can be reduced to six shearlike matrices with three lens matrix. In Sec. IV, we discuss other areas of optical sciences where the shear representation of the group  $Sp(2)$  may serve useful purposes. We also discuss a possible extension of the  $ABCD$  matrix to a complex representation, which will enlarge the group  $Sp(2)$  to a larger group.

Even though we are dealing with real  $2 \times 2$  matrices, we are using mathematical operations not too familiar in optics. In Appendix A, we show that there are two different ways of representing the  $Sp(2)$  group. One is a conventional representation combining squeeze and rotations. The other is to use two shears and one squeeze. It is shown that these are equivalent. In Appendix B, we use the Bargmann decomposition [9] to prove that the most general form of the  $ABCD$  matrix can be written as a product of one symmetric matrix and one antisymmetric matrix.

## II. DECOMPOSITIONS AND RECOMPOSITIONS

In this paper we are interested in writing the most general form of the matrix  $G$  of Eq. (1.3) as a chain of the shear matrices. It is shown in Appendix A that this is possible in terms of the generators of the  $Sp(2)$  group. Sudarshan *et al.*

[7] studied this problem in connection with para-axial lens optics. Their approach was of course correct, however they concluded that the complete calculation is "tedious."

We propose to complete this well-defined calculation by decomposing the matrix  $G$  into one symmetric matrix and one orthogonal matrix. For this purpose, let us consider the most general form of an  $Sp(2)$  matrix by referring to Appendix B. It is shown that the matrix  $G$  can be written as

$$G = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix}, \quad (2.1)$$

where the three free parameters are  $\phi$ ,  $\eta$ , and  $\lambda$ . These matrices are generated from the squeeze representation of  $Sp(2)$  given in Eq. (A6). The real numbers  $A$ ,  $B$ ,  $C$ , and  $D$  in Eq. (1.3) can be written in terms of these three parameters. Conversely, the parameters  $\phi$ ,  $\eta$ , and  $\lambda$  can be written in terms of  $A$ ,  $B$ ,  $C$ , and  $D$ , with the condition that  $AD - BC = 1$ . This matrix is written in terms of squeeze and rotation matrices. We write the last matrix of Eq. (2.1) as

$$\begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (2.2)$$

with  $\lambda = \theta - \phi$ . Instead of  $\lambda$ ,  $\theta$  becomes an independent parameter.

The matrix  $G$  can now be written as two matrices, one symmetric and the other orthogonal,

$$G = SR, \quad (2.3)$$

with

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (2.4)$$

The symmetric matrix  $S$  takes the form [2]

$$S = \begin{pmatrix} \cosh \eta + (\sinh \eta) \cos(2\phi) & (\sinh \eta) \sin(2\phi) \\ (\sinh \eta) \sin(2\phi) & \cosh \eta - (\sinh \eta) \cos(2\phi) \end{pmatrix}. \quad (2.5)$$

Our procedure is to write  $S$  and  $R$  separately as shear chains. Let us first consider the rotation matrix.

In terms of the shears, the rotation matrix  $R$  can be written as [10].

$$R = \begin{pmatrix} 1 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin \theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix}. \quad (2.6)$$

This expression is in the form of  $TLT$ , but it can also be written in the form of  $LTL$ . If we take the transpose and change the sign of  $\theta$ ,  $R$  becomes

$$R' = \begin{pmatrix} 1 & 0 \\ \tan(\theta/2) & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sin \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tan(\theta/2) & 1 \end{pmatrix}. \quad (2.7)$$

Both  $R$  and  $R'$  are the same matrix, but are decomposed in different ways.

As for the two-parameter symmetric matrix of Eq. (2.5), we start with a symmetric  $LTLT$  form

$$S = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad (2.8)$$

which can be combined into one symmetric matrix:

$$S = \begin{pmatrix} 1+a^2 & b(1+a^2)+a \\ b(1+a^2)+a & 1+2ab+b^2(1+a^2) \end{pmatrix}. \quad (2.9)$$

By comparing Eqs. (2.5) and (2.9), we can compute the parameters  $a$  and  $b$  in terms of  $\eta$  and  $\phi$ . The result is

$$a = \pm \sqrt{(\cosh \eta - 1) + (\sinh \eta) \cos(2\phi)},$$

$$b = \frac{(\sinh \eta) \sin(2\phi) \mp \sqrt{(\cosh \eta - 1) + (\sinh \eta) \cos(2\phi)}}{\cosh \eta + (\sinh \eta) \cos(2\phi)}. \quad (2.10)$$

This matrix can also be written in a  $TLTL$  form:

$$S' = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix} \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b' & 1 \end{pmatrix}. \quad (2.11)$$

Then the parameters  $a'$  and  $b'$  are

$$a' = \pm \sqrt{(\cosh \eta - 1) - (\sinh \eta) \cos(2\phi)},$$

$$b' = \frac{(\sinh \eta) \sin(2\phi) \mp \sqrt{(\cosh \eta - 1) - (\sinh \eta) \cos(2\phi)}}{\cosh \eta - (\sinh \eta) \cos(2\phi)}. \quad (2.12)$$

The difference between the two sets of parameters  $ab$  and  $a'b'$  is the sign of the parameter  $\eta$ . This sign change means that the squeeze operation is in a direction perpendicular to the original direction. In choosing  $ab$  or  $a'b'$ , we will also have to take care that the sign of the quantity inside the square root is positive. If  $\cos(2\phi)$  is sufficiently small, both sets are acceptable. On the other hand, if the absolute value of  $(\sinh \eta) \cos(2\phi)$  is greater than  $(\cosh \eta - 1)$ , only one of the sets,  $ab$  or  $a'b'$ , is valid.

We can now combine the  $S$  and  $R$  matrices in order to construct the  $ABCD$  matrix. In so doing, we can reduce the number of matrices by one:

$$SR = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & b - \tan(\theta/2) \\ 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 \\ \sin \theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix}. \quad (2.13)$$

We can also combine making the product  $S'R'$ . The result is

$$\begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix} \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b' + \tan(\theta/2) & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & -\sin \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tan(\theta/2) & 1 \end{pmatrix}. \quad (2.14)$$

For the combination  $SR$  of Eq. (2.13), two adjoining  $T$  matrices were combined into one  $T$  matrix. Similarly, two  $L$  matrices were combined into one for the  $S'R'$  combination of Eq. (2.14).

In both cases, there are six matrices, consisting of three  $T$  matrices and three  $L$  matrices. This is the minimum number of shear matrices needed for the most general form for the  $ABCD$  matrix with three independent parameters.

### III. PARA-AXIAL LENS OPTICS

So far, we have been investigating the possibilities of representing the  $ABCD$  matrices in terms of the two shear matrices. Indeed, this  $ABCD$  matrix has a deep root in ray optics [6].

In para-axial lens optics, the lens and translation matrices take the forms

$$L = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad (3.1)$$

respectively. In Sec. I, this was what we had in mind when we defined the shear matrices of  $L$  and  $T$  types. These matrices are applicable to the two-dimensional space of

$$\begin{pmatrix} y \\ m \end{pmatrix}, \quad (3.2)$$

where  $y$  measures the height of the ray, while  $m$  is the slope of the ray.

The one-lens system consists of a  $TLT$  chain. The two-lens system can be written as  $TLTLT$ . If we add more lenses, the chain becomes longer. However, the net result is one  $ABCD$  matrix with three independent parameters. In Sec. II, we asked the question of how many  $L$  and  $T$  matrices are needed to represent the most general form of the  $ABCD$  matrix. Our conclusion was that six matrices, with three lens matrices, are needed. The chain can be either  $LTLTLT$  or  $TLTLTL$ . In either case, three lenses are required. This conclusion was obtained earlier by Sudarshan *et al.* in 1985 [7]. In this paper, using the decomposition technique derived from the Bargmann decomposition, we were able to compute the parameter of each shear matrix in terms of the three parameters of the  $ABCD$  matrix.

In para-axial optics, we often encounter special forms of the  $ABCD$  matrix. For instance, the matrix of the form of Eq. (A4) is for pure magnification [11]. This is a special case of the decomposition given for  $S$  and  $S'$  in Eqs. (2.9) and (2.11) respectively, with  $\phi=0$ . However, if  $\eta$  is positive, the set  $a'b'$  is not acceptable because the quantity in the square root in Eq. (2.12) becomes negative. For the  $ab$  set, the parameters are related by

$$a = \pm (e^\eta - 1)^{1/2}, \quad b = \mp e^{-\eta} (e^\eta - 1)^{1/2}. \quad (3.3)$$

The decomposition of the  $LTLT$  type is given in Eq. (2.8).

We often encounter the triangular matrices of the form [12]

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}. \quad (3.4)$$

However, from the condition that their determinant be 1, these matrices take the form

$$\begin{pmatrix} e^\eta & B \\ 0 & e^{-\eta} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^\eta & 0 \\ C & e^{-\eta} \end{pmatrix}. \quad (3.5)$$

The first and second matrices are used for focal and telescope conditions, respectively. We call them the matrices of  $B$  and  $C$  types, respectively. The question then is how many shear matrices are needed to represent the most general form of these matrices. The triangular matrix of Eq. (3.4) is discussed frequently in the literature [11,12]. In the present paper, we are interested in using only shear matrices as elements of decomposition.

Let us consider the  $B$  type. It can be constructed either in the form

$$\begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} 1 & e^{-\eta}B \\ 0 & 1 \end{pmatrix} \quad (3.6)$$

or

$$\begin{pmatrix} 1 & e^\eta B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix}. \quad (3.7)$$

The number of matrices in the chain can be either four or five. We can reach a similar conclusion for the matrix of the  $C$  type.

#### IV. OTHER AREAS OF OPTICAL SCIENCE

We write the  $ABCD$  matrix for the ray transfer matrix [11]. There are many ray transfers in optics other than para-axial lens optics. For instance, a laser resonator with spherical mirrors is exactly like para-axial lens optics if the radius of the mirror is sufficiently large [13].

If wave fronts with phase are taken into account, or for Gaussian beams, the elements of the  $ABCD$  matrix become complex [14,15]. In this case, the matrix operation can sometimes be written as

$$w' = \frac{Aw + B}{Cw + D}, \quad (4.1)$$

where  $w$  is a complex number with two real parameters. This is precisely the bilinear representation of the six-parameter Lorentz group [9]. This bilinear representation was discussed in detail for polarization optics by Han *et al.* [16]. This form of representation is also useful in laser mode-locking and optical pulse transmission [15].

The bilinear form of Eq. (4.1) is equivalent to the matrix transformation [16]

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (4.2)$$

with

$$w = \frac{v_1}{v_2}. \quad (4.3)$$

This bilinear representation deals only with the ratio of the first component to the second in the column vector to which the  $ABCD$  matrix is applicable. In polarization optics, for instance,  $v_1$  and  $v_2$  correspond to the two orthogonal elements of polarization.

Indeed, this six-parameter group can accommodate a wide spectrum of optics and other sciences. Recently, the  $2 \times 2$  Jones matrix and  $4 \times 4$  Mueller matrix were shown to be  $2 \times 2$  and  $4 \times 4$  representations of the Lorentz group [1]. Also recently, Monzón and Sánchez showed that multilayer optics could serve as an analog computer for special relativity [5]. More recently, it was noted that two-beam interferometers can also be formulated in terms of the Lorentz group [17].

#### V. CONCLUDING REMARKS

The Lorentz group was introduced to physics as a mathematical device to deal with Lorentz transformations in special relativity. However, this group is becoming the major language in optical sciences. With the appearance of squeezed states as two-photon coherent states [18], the Lorentz group was recognized as the theoretical backbone of coherent states as well as generalized coherent states [4].

In their recent paper [2], Han *et al.* studied in detail possible optical devices which produce the shear matrices of Eq. (1.1). This effect is due to the mathematical identity called ‘‘Iwasawa decomposition’’ [19,20]. The shear matrices of Eq. (1.1) are products of Iwasawa decompositions. Since we used those shear matrices to produce the most general form of  $2 \times 2$  unimodular matrices with three real parameters, in this paper we are performing an inverse process of the Iwasawa decomposition.

It should be noted that the decomposition we used in this paper has a specific purpose. If purposes are different, different forms of decomposition may be employed. For instance, decomposition of the  $ABCD$  matrix into shear, squeeze, and rotation matrices could serve useful purposes for canonical operator representations [12,21]. The amount of calculation seems to depend on the choice of decomposition.

Group theory in the past was understood as an abstract mathematics. In this paper, we have seen that it can be used as a calculational tool.

#### APPENDIX A: SQUEEZE AND SHEAR REPRESENTATIONS OF THE $Sp(2)$ GROUP

The  $ABCD$  matrix is a shear representation of the group  $Sp(2)$ . The shear matrices of Eq. (1.1) can be written as

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \exp(-isX_1), \quad (A1)$$

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \exp(-iuX_2),$$

with

$$X_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad (A2)$$

which serve as the generators. If we introduce a third matrix

$$X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (\text{A3})$$

it generates squeeze transformations:

$$\exp(-i\eta X_3) = \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix}. \quad (\text{A4})$$

The matrices  $X_1$ ,  $X_2$ , and  $X_3$  form the following closed set of commutation relations:

$$\begin{aligned} [X_1, X_2] &= iX_3, & [X_1, X_3] &= -2iX_1, \\ [X_2, X_3] &= 2iX_2. \end{aligned} \quad (\text{A5})$$

The generators  $X_1$  and  $X_2$  produce the third generator  $X_3$ . Then these three generators form a closed set of commutation relations for the  $\text{Sp}(2)$  group [3,10,22].

The  $\text{Sp}(2)$  group can be generated by two seemingly different sets of generators, namely the shear-squeeze generators of Eqs. (A2) and (A3) and the squeeze-rotation generators, which are conventionally expressed as

$$\begin{aligned} B_1 &= \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & B_2 &= \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ J &= \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \quad (\text{A6})$$

when applied to a two-dimensional  $xy$  space. The  $J$  matrix generates rotations around the origin while  $B_1$  and  $B_2$  generate squeezes along the  $xy$  axes and along axes rotated by  $45^\circ$ , respectively. It is clear that one representation can be transformed into the other at the level of generators. The generators of Eqs. (A2) and (A3) can be written as

$$X_1 = B_2 - J, \quad X_2 = B_2 + J, \quad X_3 = 2B_1, \quad (\text{A7})$$

where  $J$ ,  $B_1$ , and  $B_2$  are given in Eq. (A6).

## APPENDIX B: BARGMANN DECOMPOSITION

In his 1947 paper [9], Bargmann considered

$$W = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad (\text{B1})$$

with  $\alpha\alpha^* - \beta\beta^* = 1$ . There are three independent parameters. Bargmann then observed that  $\alpha$  and  $\beta$  can be written as

$$\alpha = (\cosh \eta) e^{-i(\phi+\lambda)}, \quad \beta = (\sinh \eta) e^{-i(\phi-\lambda)}. \quad (\text{B2})$$

Then  $W$  can be decomposed into

$$W = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} e^{-i\lambda} & 0 \\ 0 & e^{i\lambda} \end{pmatrix}. \quad (\text{B3})$$

In order to transform the above expression into the decomposition of Eq. (2.1), we take the conjugate of each of the matrices with

$$C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (\text{B4})$$

Then  $C_1 W C_1^{-1}$  leads to

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix}. \quad (\text{B5})$$

We can then take another conjugate with

$$C_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (\text{B6})$$

Then the conjugate  $C_2 C_1 W C_1^{-1} C_2^{-1}$  becomes

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix}. \quad (\text{B7})$$

This expression is the same as the decomposition given in Eq. (2.1).

The combined effect of  $C_2 C_1$  is

$$C_2 C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}. \quad (\text{B8})$$

If we take the conjugate of the matrix  $W$  of Eq. (B1) using the above matrix, the elements of the  $ABCD$  matrix become

$$\begin{aligned} A &= \frac{1}{2}(\alpha + \alpha^* + \beta + \beta^*), \\ B &= \frac{-i}{2}(\alpha - \alpha^* + \beta - \beta^*), \\ C &= \frac{i}{2}(\alpha - \alpha^* - \beta + \beta^*), \end{aligned} \quad (\text{B9})$$

$$D = \frac{1}{2}(\alpha + \alpha^* - \beta - \beta^*).$$

We can see from this expression that all the elements in the  $ABCD$  matrix are real numbers. Indeed, the  $\alpha\beta$  representation of Eq. (B1) is equivalent to the  $ABCD$  representation, whose components can be written as

$$\begin{aligned} A &= (\cosh \eta) \cos(\phi + \lambda) + (\sinh \eta) \cos(\phi - \lambda), \\ B &= -(\cosh \eta) \sin(\phi + \lambda) - (\sinh \eta) \sin(\phi - \lambda), \\ C &= (\cosh \eta) \sin(\phi + \lambda) - (\sinh \eta) \sin(\phi - \lambda), \\ D &= (\cosh \eta) \cos(\phi + \lambda) - (\sinh \eta) \cos(\phi - \lambda). \end{aligned} \quad (\text{B10})$$

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